This is the second of the two real equations that arise from Eq. (2). On the other hand, from the Kutta condition (1)

$$U - K/2\tau = -K(\sigma^2 - 3\tau^2)/2\tau(\sigma^2 + \tau^2)$$

$$= -Ka^2/2\tau(\sigma^2 + \tau^2), \text{ by (4)} \qquad (6)$$

Eqs. (5) and (6) are incompatible for finite values of σ and τ, so we conclude that no solution exists. Since Eqs. (5) and (6) differ only by a factor of 2, it seems likely that both Riabouchinsky and Coe lost this factor somewhere, leading them to think that the Kutta condition was satisfied automatically. Of course, from the equation $2 = 1$ anything follows, including the two different loci quoted in Refs. 4 and 5.

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Ion Density Measurements with an Electric Probe

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THIS Note reports on the application of an electric probe at large negative potential to measuring the ion density in a flowing, slightly ionized gas. The experimental conditions were such that an ion sheath formed on the ion collecting electrode. The viscous boundary layer was thin compared with the ion sheath. This allowed use of the theory of Ref. 1 for evaluation of the data. Although the theory was developed for a flat plate and the probe used was cylindrical, the theory should be quite good when the sheath is thinner than the transverse electrode dimension (i.e., for $n > 10^{10}$ cm^{-3} at our conditions).

The dimensions of the probe used for these experiments are shown in Fig. 1. The Pyrex body, nominally 6 mm i.d., was hot-formed over a mandrel to provide the diverging cross section. The shape was chosen to compensate approximately for boundary-layer growth in the flow behind a Mach 9 shock into air at an initial pressure of 0.3 torr. For the conditions of interest here, the ionization relaxation time is much longer than the transit time through the probe, and hence a slight expansion or contraction of the flow (barring choking or shock formation) will not affect the ionization level of the stream. The probe leading edge was sharpened after forming. Silver paint electrodes covering 90° quadrants and an external electrostatic shield were applied by brush and fired. Fine stranded leads were soldered to the electrodes which were carried around the fire-polished back edge of the probe body so that all connections could be made on the outside surface for minimal flow disturbance. The completed probe was then coated externally with corona lacquer to insulate it from the plasma and mounted in a tapered, anodized aluminum holder. Leads were epoxied to the inside wall of the holder and brought back 12 in. to a mounting plate for connection to coaxial cable. The entire assembly was mounted on a 2-in. diam hollow rod extending 3 ft from the end of the shock tube. This avoided generation of flow disturbances or a reflected shock inside the probe during the test time.

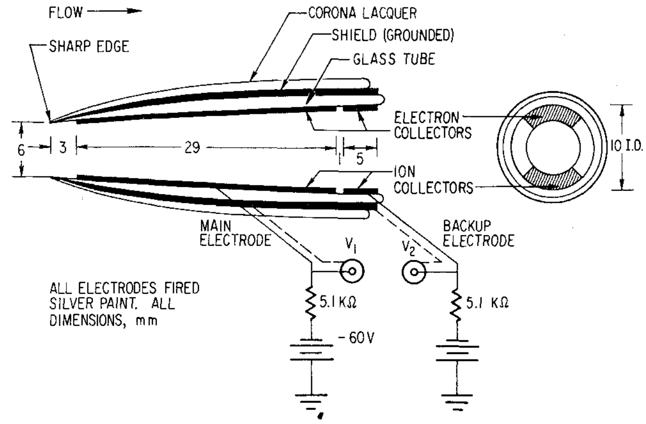

Fig. 1 Geometry of electric probe.

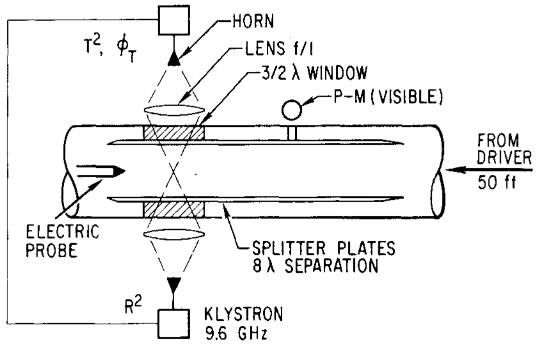

Fig. 2 Instrumentation of 17 in. shock tube.

A microwave probe[2] was used to determine the electron density independently. The overall experimental arrangement is shown in Fig. 2. The 17-in. shock tube was fitted with a rectangular box with sharpened leading edge to isolate a plane slab of gas. A focused microwave beam at 9.6 GHz ($\lambda_0 = 3.13$ cm) passes through resonant windows and through the gas sample. It has been shown by Primich et al.[3] that an f/1 lens system with tapered illumination (as provided by the 15 db horn pattern) has optimal resolution of approximately 3/2 wavelength between 3 db points transverse to the beam axis. It has also been shown[4,5] that the focused aperture system generates a very nearly plane wave front for a distance of $4\lambda_0$ on either side of the focus; this, and a desire for maximum sensitivity, determined the $8\lambda_0$ width of the test slab. To avoid distortion and refraction of the wave front, the focused probe should be operated in a highly underdense plasma. For all of our experiments, the collision ratio ν_c/ω was less than 0.1 and

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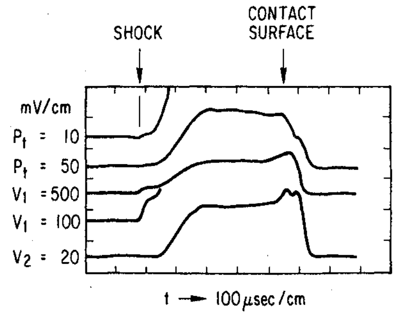

Fig. 3 Oscillograph records of microwave and electric probe output. $V_s = 2.71$ mm/μs, $p_1 = 0.3$ torr, 70% air/30% argon.

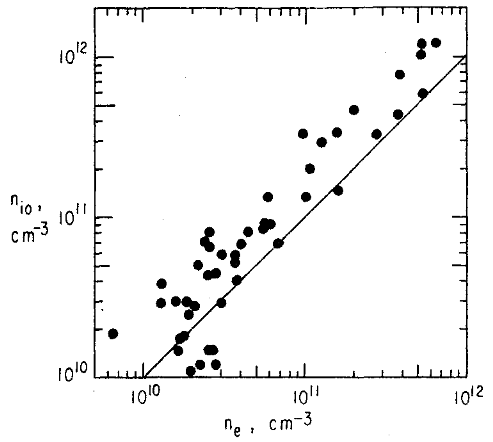

Fig. 4 Measured freestream ion density vs measured electron density.

the probe could be operated up to almost the critical density of 1.2×10^{12} cm^{-3} with good accuracy. The proper operation of the microwave system was verified by calibration on dielectric slugs of known properties to an accuracy of 20% or better. Plane wave propagation theory is justified for this geometry, and comparison of measured transmission coefficient and phase shift with calculated curves permits determination of electron density and collision frequency to accuracy of ±30% and ± a factor of 2, respectively, due to all error sources. Corrections for boundary-layer growth[6] are negligible.

During its early development,[11] the ion probe served only as a backup to the primary microwave electron density probe. The advent of a more comprehensive theoretical foundation [1,13,16] has prompted the publication of this more complete analysis of ion probe performance. These experiments were run as part of a more comprehensive study of air chemistry at elevated temperature.[7] The most important conclusion for present purposes is that the air ionization mechanism established by Lin and Teare[8] at 8000 K also holds for 3000 K; thus, the ion of interest is NO^+, and its mobility will control the sheath conduction process on the electric probe.

The 17-in. stainless steel shock tube[9] has been described in earlier nonequilibrium flow studies.[10] Its 3-in. driver is connected to the test section by a 15° conical transition. Cold He or H_2 driver at 5000 psi or 2000 psi max, and scribed aluminum diaphragms are used. Shock arrival at successive instrument stations was determined with thin film heat transfer gages. The runs made for the present study covered a range of shock velocity from 2.7-3.45 km/sec at initial pressures of 0.075-1.20 torr. Both air and mixtures of air and argon were used as test gases.

The ion probe was connected electrically as shown in Fig. 1. The electron collectors were simply grounded to the shock tube, and a potential of -60 V was applied through a 5$k\Omega$ resistor to the primary and secondary ion collectors. A representative oscilloscope trace showing the output signals

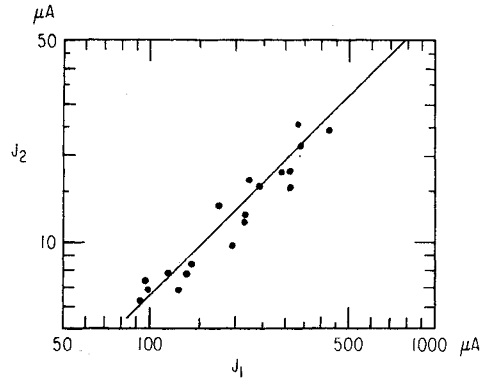

Fig. 5 Current J_2 from secondary electrode as function of current J_1 from primary electrode. Line is approximate theoretical result.

V_1 and V_2 from the probe, as well as the transmitted phase bridge output P_t from the microwave probe, appears in Fig. 3. A calculation of the sheath thickness indicated that it would be smaller than the probe diameter at $n_i > 2 \times 10^9$ cm^{-3}. The secondary collector does indeed begin to show current at about this electron density. At lower densities, the probe simply collects all ions convected into its entrance area.[11]

In applying the theory of Ref. 1 for the ion sheath to the evaluation of the freestream ion density, the ion mobility K must be known. No measurements of ion mobility are available for air or argon at 3000 K. An approximate value for K was obtained by using the equation relating K to the dielectric constant of the neutral gas (Eq. 1.7 of Ref. 12). This yielded $K = 2.5\ \rho_0/\rho$ cm^2/V sec for both air and argon; here ρ_0 is the gas density at 0°C and 760 torr, and ρ is the actual gas density.

Since the ion density inferred is approximately proportional to $K^{1/3}$ [cf. Eq. (11) of Ref. 13], the final results are not very dependent on the value taken for K. It is believed that the value cited is sufficiently accurate for present purposes. In order to determine various other quantities needed in the data reduction, use was made of the tables and charts given in Refs. 14 and 15.

The results obtained are shown in Fig. 4, where the measured ion density is plotted as a function of the measured electron density. It is seen that the agreement generally is quite good. On the average, the data obtained for the ion density are about 50% larger than those for the electron density. It is likely that the difference arises because some ion production occurred in the electric sheath.[16] While the theory of Ref. 16 can be used to correct for this effect, this was not considered worthwhile for the present case in view of the relatively large scatter in the data obtained (extreme points differ from the average by a factor of about 2).

According to theory,[1,13] the current J_2 collected on the backup electrode should be approximately proportional to the current J_1 collected on the main one. For the electrode lengths used (ignoring the 1-mm gap), the ratio J_2/J_1 should be approximately $(35^{1/4} - 29^{1/4})/(29)^{1/4} = 0.048$. In Fig. 5, J_2 is plotted as function of J_1, together with the approximate theoretical result. The data shown for J_2 are those available; in a number of runs, only J_1 was obtained. It should be noted that in one set of 5 consecutive runs the J_2 traces were off-scale, and the data obtained for J_1 in these runs yielded the 5 points considerably below the line of Fig. 4. Apart from this pecularity (which we strongly suspect is an error in gain settings), we conclude that for the conditions of these experiments, the electric probe provides a satisfactory measurement of ion density.

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Cascade with Subsonic Leading-Edge Locus

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Introduction

IN modern aircraft engine technology there is considerable interest in the problem of the unsteady supersonic cascade with subsonic axial velocity. In this Note, we consider a two-dimensional oscillating cascade with a subsonic leading-edge locus in a supersonic flow which is uniform far upstream. It is assumed that the blades have small thickness and camber and are undergoing small amplitude harmonic oscillations. Kurosaka[1] has obtained a low frequency analytical solution to this problem and Verdon[2] has obtained a finite-difference solution. This Note will show that the problem can be reduced solving a functional integral equation and give a representation of the kernel function which is useful for computation. The detailed structure of this equation is explicitly displayed.

Derivation of Integral Equation

The blades, which are assumed to have small thickness and camber, are undergoing small amplitude harmonic oscillations. We suppose that all lengths are nondimensionalized by the half-blade chord $c/2$; the time t is nondimensionalized with respect to $c/2$ divided by the freestream velocity U; the pressure fluctuation p is nondimensionalized by ρ_0, the freestream density, times U^2; and the upwash velocity v is nondimensionalized by U. Let a denote the freestream speed of sound. Then the pressure fluctuation is governed by the equation (see Fig. 1)

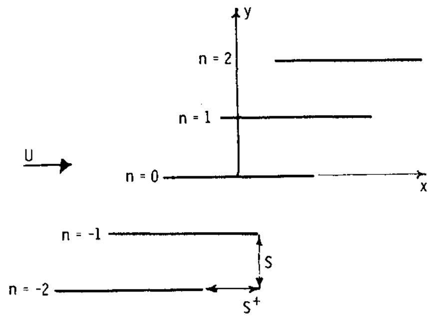

Fig. 1 Dimensionless cascade configuration.

$$\frac{\partial^2\psi}{\partial y^2} - \beta^2 \frac{\partial^2\psi}{\partial x^2} - \beta^2 k^2 \psi = 0 \tag{1}$$

where

$$\psi \equiv p e^{i(\omega t - Mkx)} \tag{2}$$

$\beta^2 = M^2 - 1$; $k = \omega M/\beta^2$; $M = U/a$ is the freestream Mach number; and the upwash velocity, v, related to the pressure by

$$e^{i\beta^2 kx/M} \frac{\partial}{\partial x} (e^{-i\beta^2 kx/M} V) = -e^{iMkx} \frac{\partial\psi}{\partial y} \tag{3}$$

where $V = v \exp[i\omega t]$.

The upwash velocity on the nth blade is assumed to differ from that of the 0th blade by only a phase factor so that

$$V(x+ns^\dagger, ns) = e^{in\sigma} V(x,0) \quad \text{for } |x|<1, n=0, \pm 1, \pm 2 \tag{4}$$

where σ is the interblade phase angle and the upwash velocity on the 0th blade is related to its displacement $W_0 e^{-i\omega t}$ by

$$V(x,0) = \left[-i\omega + \frac{\partial}{\partial x}\right] W_0(x) \quad \text{for } |x|<1$$

As is usual, suppose for convenience that the frequency has a small positive imaginary part which we shall set equal to zero at the end of the analysis. Then $k = k_r + i\epsilon$ for $0 < \epsilon \ll 1$ and the outgoing wave boundary condition at infinity is now replaced by a boundedness condition.

Since Eq. (1) possesses the separation of variables solution exp. $\{-[i(\alpha x - \beta\gamma y)]\}$, where

$$\gamma \equiv \sqrt{\alpha^2 - k^2}$$

the boundary condition Eq. (4) suggests that we seek a solution in the form of the super-position

$$\psi = \sum_{n=-\infty}^{\infty} \psi_n \tag{5}$$

where

$$\psi_n \equiv (\operatorname{sgn} y_n)/2 \int_{-\infty+i\delta}^{\infty+i\delta} f_n(\alpha) e^{-i(\alpha x_n - \beta\gamma|y_n|)} d\alpha \tag{6}$$

we have put (see Fig. 1) $x_n = x - ns^\dagger$; $y_n = y - ns$ for $n = 0, \pm 1, \pm 2, \ldots$, and in order to insure that the solution remains boun-

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